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Stable Gravity Wave of Arbitrary Amplitude in Finite Depth

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A combination and modification of two existing methods, which involves balancing static and dynamic pressure differences between points along the surface and conserving mass through cross sections below the surface in the reference frame moving with the phase velocity, is applied to surface gravity waves of arbitrary amplitude in water of finite depth. For a given still water depth and wave height the method determines in closed form the phase velocity, wavelength, and wave profile of the stable wave. The main assumption is that the horizontal component of the fluid velocity be independent of depth. The motion is not assumed to be irrotational. The wavelength of the stable wave is found to be about 3.6 times the still water depth for infinitesimal amplitude, and at finite amplitude the wavelength decreases as the amplitude increases. Therefore, shallow water waves are concluded to be unstable even at infinitesimal amplitude, for which the assumption is accurate. Previously it has been argued that only at finite amplitude will shallow water waves change form as they propagate. The wave profile is found to be sinusoidal for infinitesimal amplitude and to be asymmetric at finite amplitude, the crests being higher and narrower and the troughs shallower and broader. These results are consistent with well-known theoretical work and laboratory measurements.

1. INTRODUCTION

A combination and modification of two existing methods is applied to surface gravity waves of arbitrary amplitude in water of finite constant depth in order to study the properties of stable waves, such as the phase velocity, wavelength, and wave profile. The vast majority of theoretical work on surface gravity waves begins with the assumption of irrotational motion (potential flow), but the method given below is free of this assumption.

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Also, available theories of finite-amplitude waves almost invariably invoke some form of perturbation expansion, with the accompanying disadvantages that the convergence of the expansion is usually not proved and the solution obtained is normally not checked by independent means. In contrast to this, the present method allows the solutions for the wave properties to be obtained in closed form without the use of perturbation expansions. The phase velocity is given by an algebraic expression, and the wavelength and surface elevation are determined by elliptic functions.

The method used here is based on only minor adjustments to two early models, one contained in a little-known paper by Einstein (1916) and the other in a well-known paper by Lord Rayleigh (1876). Nevertheless, the following unexpected result is obtained: for a given still water depth and a given wave height there is only one stable wave, and furthermore the wavelength of this wave is comparable to the depth. As a consequence of this result, it is inferred that true shallow water waves (wavelength $\gg$ water depth) must be unstable even for infinitestimal amplitude (amplitude $\ll$ water depth). Previously it was argued by Airy, who used a different analysis technique (Lamb, 1932, p. 278), that only at finite amplitude would a shallow water wave change form as it propagates.

On the other hand, the result for the wave profile agrees qualitatively with past results, both theoretical and experimental. The wave profile is found to be sinusoidal at infinitesimal amplitude and to be asymmetric at finite amplitudes. The asymmetry is characterized by shallower and broader troughs and higher and narrower crests.

The method proceeds in two parts as follows. First, the phase velocity is computed by adapting Einstein's (1916) model to water of constant finite depth. Second, by combining a small variation of a method due to Lord Rayleigh (1876) with the results of the first part, the wavelength and wave profile are calculated.

The central assumption on which the results are based is that the horizontal components of the fluid velocity should be independent of depth. This assumption has often been made in past theories of shallow water waves and it is accurate for such waves at infinitesimal amplitude, which, however, are found below to be unstable.

2. PHASE VELOCITY

The phase velocity is calculated with the help of a useful physical model due to Einstein (1916). Consider a frictionless and incompressible fluid in a channel. The channel has a flat rigid bottom, a rigid corrugated top, and is uniform in the direction perpendicular to the cross section illustrated schematically in Figure 1.

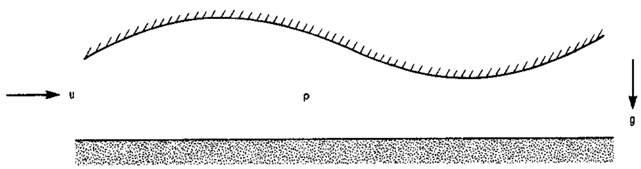

Fig. 1. Fluid of constant density ρ entering a channel with steady horizontal speed u. The acceleration of gravity g acts down. The channel has a flat rigid bottom and a rigid corrugated top. The "wavelength" of the corrugations is comparable to the mean water depth. The channel is uniform in the direction perpendicular to the paper. The fluid entirely fills the channel. Friction is neglected.

When fluid fills the channel but is not flowing, there is a static pressure difference between crest and trough on the top surface due to the acceleration of gravity. The static pressure is greater at the trough than at the crest because gravity acts downward.

When fluid flows steadily through the channel, conservation of mass states that in unit time the same amount of fluid must flow through all cross sections. The fluid flows faster under the trough than under the crest because the vertical cross section is smaller below the trough than below the crest.

If the acceleration of gravity is not acting and fluid flows steadily through the channel, Bernoulli's principle states that the pressure is least where the speed is greatest (and vice versa), or

$$p = \text{const} - \tfrac{1}{2}\rho u^2 \qquad (2.1)$$

where p is the pressure, ρ the density, and u the speed of the fluid, and (2.1) holds along a streamline. Then there will be a dynamic pressure difference between crest and trough on the top surface. According to (2.1), the dynamic pressure is less at the trough than at the crest because the fluid speed is greater at the trough than at the crest, as required by conservation of mass.

Finally, let the fluid flow in the channel and let the acceleration of gravity act downward. Then there will be both static and dynamic pressure differences between crest and trough, and they will be oppositely directed. Therefore, it is possible to balance the two oppositely directed pressure difference between crest and trough on the top surface by the right choice of flow rate. When that balance occurs, the top surface of the channel can be taken away without change in the stationary "wavy" surface of the fluid. The usual picture of waves propagating over the flat bottom, viewing the fluid in the reference frame that moves with the phase velocity of the wave, is equivalent to the balanced channel flow. This is the essence of Einstein's (1916) method, which we will now use for calculating the phase velocity.

The static pressure difference at the surface between crest and trough, Δp_S, is

$$\Delta p_S = \rho g H \tag{2.2}$$

where g is the acceleration of gravity and H is the wave height (Figure 2).

In order to compute the dynamic pressure difference at the surface between crest and trough, the average height of the crest and trough above the bottom, $\bar{h}$, is introduced, which is to be distinguished from the still water depth h. The crest is then at the height $\bar{h}+H/2$ and the trough is at $\bar{h}-H/2$. Next, the horizontal fluid velocities at the trough and crest are taken to be $u+\Delta u_T$ and $u-\Delta u_C$, respectively. Then the dynamic pressure difference between crest and trough, Δp_D, becomes

$$\Delta p_D = \tfrac{1}{2}\rho[(u+\Delta u_T)^2-(u-\Delta u_C)^2] \tag{2.3}$$

Equation (2.3) comes from applying (2.1) to the surface streamline.

Conservation of mass between trough and crest (without the constant density) is

$$(u+\Delta u_T)\left(\bar{h}-\frac{H}{2}\right)=(u-\Delta u_C)\left(\bar{h}+\frac{H}{2}\right)=uh \tag{2.4}$$

where uh relates to the mass flux at a position with no waves. From (2.4) follows

$$\begin{aligned}\frac{\Delta u_T}{u}&=\frac{H/2\bar{h}+1-h/\bar{h}}{1-H/2\bar{h}}\\ \frac{\Delta u_C}{u}&=\frac{H/2\bar{h}-1+h/\bar{h}}{1+H/2\bar{h}}\end{aligned} \tag{2.5}$$

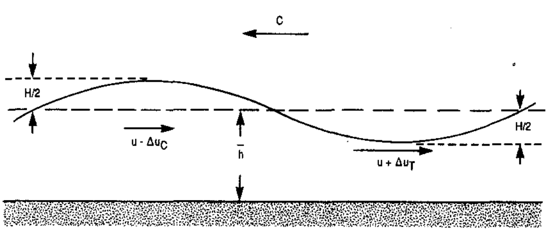

Fig. 2. Surface gravity wave viewed in the reference frame moving to the left with the phase velocity of magnitude c. The wave form remains stationary with respect to the observer and the fluid flows beneath it steadily to the right. The horizontal fluid speeds beneath crest and trough are $u-\Delta u_C$ and $u+\Delta u_T$, respectively, and the phase speed of the wave is given by $c=u$. The average depth of crest and trough is $\bar{h}$, the crest and trough are a distance $H/2$ above and below $\bar{h}$, respectively, and H is the wave height. The wavelength is comparable to the water depth. The figure is schematic.

The central assumption has now been used in conserving mass in (2.4): the horizontal velocity components are taken independent of depth below the crest and trough.

Eliminate Δu_T and Δu_C in equation (2.3) with either (2.4) or (2.5) to get

$$\Delta p_D = \frac{\rho u^2 (H/\bar{h})(h/\bar{h})^2}{[1-(H/2\bar{h})^2]^2} \tag{2.6}$$

Now balancing the static and dynamic pressure differences $\Delta p_S = \Delta p_D$ between (2.2) and (2.6) yields

$$u^2 = gh\left\{\left[1-\left(\frac{H}{2\bar{h}}\right)^2\right]^2\left(\frac{\bar{h}}{h}\right)^3\right\} \tag{2.7}$$

It will be shown below that for infinitesimal amplitude, $(H/2\bar{h})^2 \ll 1$, $\bar{h} \to h$ and therefore (2.7) becomes

$$u^2 = gh \tag{2.8}$$

The speed u in (2.7) and (2.8) can be interpreted to be the phase speed of the wave c by moving in the reference frame with speed u. Then (2.7) is $c^2 = gh$, the well-known formula for the speed of small-amplitude waves in shallow water due to Lagrange (1869).

The result (2.8) taken by itself seems to be a natural one in view of the main assumption used that the horizontal component of the fluid velocity be independent of depth. This assumption is often used in the theory of long waves. However, the above derivation of (2.8) is not at all familiar. A more usual way to derive (2.8) starts with irrotational waves of infinitesimal amplitude in water of arbitrary constant depth and then proceeds to the shallow water limit (e.g., Barnett and Kenyon, 1975). Here we started with finite-amplitude waves in finite depth, assumed homogeneous flow below crest and trough but not irrotational motion, and then took the infinitesimal-amplitude limit.

3. WAVELENGTH AND PROFILE

The wavelength and profile of a stable wave are calculated by balancing the static and dynamic pressure differences, and conserving mass through the cross sections, between the trough and any arbitrary point along the surface from the trough to the adjacent crest (Figure 3). This is an extension of Lord Rayleigh's (1876) method to include the vertical component of the fluid velocity.

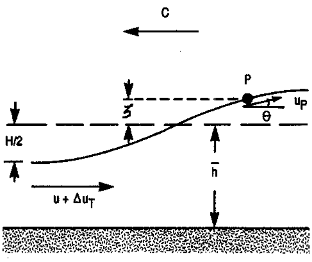

Fig. 3. Similar to Fig. 2, but illustrating the arbitrary point P on the surface and its elevation ζ relative to $\bar{h}$. The fluid velocity of magnitude u_P is tangent to the surface at P and is directed at angle θ to the horizontal.

The balance of static and dynamic pressure differences between the wave trough and an arbitrary point P on the surface is

$$g\left(\frac{H}{2}+\zeta\right)=\tfrac{1}{2}[(u+\Delta u_T)^2-u_P^2] \tag{3.1}$$

where the density has already been canceled out. The elevation ζ of the arbitrary point P is measured from the average height of crest and trough $\bar{h}$. The fluid speed tangent to the surface at P is u_P.

Conservation of mass below the trough and below P gives

$$(u+\Delta u_T)\left(\bar{h}-\frac{H}{2}\right)=u_P\cos\theta(\bar{h}+\zeta) \tag{3.2}$$

where θ is the angle between the surface tangent at P and the horizontal. In (3.2) the horizontal component of the fluid velocity has again been taken independent of depth, but the vertical velocity component has not been neglected.

Eliminate u_P between (3.1) and (3.2) to obtain

$$\cos\theta=\frac{1-(H/2\bar{h})^2}{[1+(H/2\bar{h})^2-2\zeta/\bar{h}]^{1/2}(1+\zeta/\bar{h})} \tag{3.3}$$

where u and Δu_T have been replaced by H, $\bar{h}$, and h through (2.5) and (2.7). Notice that (3.3) is independent of h.

Relating $\cos\theta$ to $\tan\theta=\partial\zeta/\partial x$ and incorporating (3.3) results in a first-order differential equation for the surface elevation ζ of a stable wave,

$$\frac{\partial\zeta}{\partial x}=\frac{1}{1-(H/2\bar{h})^2}\left\{\left[\left(\frac{H}{2\bar{h}}\right)^2-\left(\frac{\zeta}{\bar{h}}\right)^2\right]\left[3-\left(\frac{H}{2\bar{h}}\right)^2+\frac{2\zeta}{\bar{h}}\right]\right\}^{1/2} \tag{3.4}$$

as a function of the horizontal coordinate x and the given quantities H and $\bar{h}$. Equation (3.4) can be rewritten in integral form as

$$\frac{x}{\bar{h}}=(1-a^2)\int_{-a}^{y}\frac{dt}{\{[a^2-t^2][3-a^2+2t]\}^{1/2}} \tag{3.5}$$

where $a=H/2\bar{h}$, $t=\zeta/\bar{h}$, $-a\le y\le a$, and x is measured from the trough.

The wavelength λ is twice the horizontal distance between the trough ($y=-a$) and the next crest ($y=a$). From (3.5) then

$$\frac{\lambda}{\bar{h}}=4(1-a)\left[\frac{1+a}{3-a}\right]^{1/2}F\left(\frac{\pi}{2},k\right) \tag{3.6}$$

where $F(\pi/2,k)$ is the complete elliptic integral of the first kind and $k^2=4a/(1+a)(3-a)$ (Byrd and Friedman, 1971, p. 77, #235.00). For infinitesimal amplitude, $a\ll 1$, equation (3.6) reduces to

$$\frac{\lambda}{h}=\frac{2\pi}{\sqrt{3}}\approx 3.6 \tag{3.7}$$

because $\bar{h}\to h$, as will become clear shortly. For finite amplitude it can be shown from (3.6) that the wavelength decreases as the amplitude increases. Therefore, the maximum wavelength allowable for stable waves is given by (3.7), and it is only a small multiple of the still water depth.

The wave profile $\zeta(x)$ is obtained from (3.5) in the form $x(\zeta)$ for given H and $\bar{h}$ by solving the incomplete elliptic integral of the first kind for successive values of y. The wave profile turns out to be sinusoidal for infinitesimal amplitude, and therefore $\bar{h}=h$. At finite amplitude it can be shown from (3.5) that the wave profile is not quite symmetric in that the crests become a little higher and narrower and the troughs shallower and broader (consequently $\bar{h}>h$).

Once $\zeta(x)$ is found, then $\bar{h}$ can be obtained in relation to h by the requirement that over one wavelength there must be as much water above as below h, or $\int_0^\lambda \int_0^{\bar{h}+\zeta} dz\,dx=h\lambda$, giving $\bar{h}=h-(1/\lambda)\int_0^\lambda \zeta(x)\,dx$, where z measures distance along the vertical coordinate.

4. DISCUSSION

To summarize the results, we have described a direct approach for calculating wave parameters that is an evolution of some work by Einstein and Lord Rayleigh. The main assumption about a constant velocity profile with depth is generally accepted in the theory of shallow water waves. Utilizing this assumption, we found that shallow water waves of infinitesimal amplitude propagate with the well-known phase velocity $(gh)^{1/2}$, but that they are unstable. It is not possible to balance the pressure differences at the surface and conserve mass between crest and trough when the wavelength is much greater than the water depth.

We also found that the profile of a stable wave of finite amplitude has narrow crests and wide troughs, as has been calculated with a perturbation expansion by Stokes and observed experimentally. These stable waves have a well-defined relation between wavelength and water depth, but are not dispersive (the phase velocity is independent of wavelength). It is reasonable to expect that when the main assumption about the constant velocity profile is relaxed to include vertical shear, the phase velocity will depend on the vertical shear (which in turn will be related to the wavelength through the vertical fluid acceleration) and that is how dispersion will appear. Incorporating a given shear into the calculation of the phase velocity is not difficult, once we know which one to choose. (For an exponential shear, $c \sim (g\lambda)^{1/2}$.)

The real convenience of assuming that the horizontal component of the fluid velocity is independent of depth enters in Section 3 where the wavelength and surface elevation are calculated. It has not yet been found possible to make a simple estimate of how much the maximum wavelength allowable for stable waves in (3.7), for example, would change if a vertical shear were present, or to figure out how different the wave profile would be. Possibly an adaptation of the present method for numerical calculation could yield some insights here.

One consequence of the present model, in which pressure balances and mass conservation are only carried out within a single wavelength, is that it may not be necessary to view the stable wave we have calculated as an infinite wave train. Therefore the possibility exists that a similar method might apply to the solitary wave.

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